Kyiv
School of
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Conditional probabilities

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Outline

1. Conditional probability
2. Bayes' Theorem

Conditional probability – 1

General definition: under assumption that P(B) > 0 (event B has to be possible)

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Conditional probability – 2

- Useful expressions:

$$P(A) = P(A \mid \Omega)$$ for any event A

$$A \cap B = \varnothing \Rightarrow P(A \mid B) = 0$$

$$A \subset B \Rightarrow P(A \mid B) = \frac{P(A)}{P(B)}$$

$$B \subset A \Rightarrow P(A \mid B) = 1$$

Example – 1

If we randomly choose a family with two children and find out that at least one child is a boy, find the probability that both children are boys.

$$\Omega = \{\text{boy boy, boy girl, girl boy, girl girl}\}$$

- A = event that both children are boys
- B = event that at least one is a boy

Example – 2

- A = event that both children are boys
- B = event that at least one is a boy

$$\Omega = \{\text{boy boy, boy girl, girl boy, girl girl}\}$$

$$P(B) = \frac{3}{4} \quad \text{and} \quad P(A \cap B) = \frac{1}{4}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Example

- We are throwing a 6-sided die three times. Each time we have got a different number of dots. Calculate a probability that **<u>only once</u>** we get a „5” assuming that each attempt gives different number.

$$P(B)=\frac{6\cdot 5\cdot 4}{6\cdot 6\cdot 6}=\frac{120}{216}$$

$$P(A\cap B)=\frac{6\cdot 5\cdot 4-5\cdot 4\cdot 3}{6\cdot 6\cdot 6}=\frac{60}{216}$$

$$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{60}{120}=\frac{1}{2}$$

Multiplication Rule

Definition. The probability that events *A* and *B* both occur is one trial of a random experiment is

$$P(A \cap B) = P(A)P(B \mid A)$$

Example – 1

Randomly choose two different students. Find the probability they *both* completed the problems

A = event first student completed the problems

B = event second student completed the problems

	Grade		
	A or B	C or below	Totals
Completed	34	8	42
Did Not Complete	12	26	38
Totals	46	34	80

Example – 2

$$P(A) = \frac{42}{80} = \frac{21}{40} \qquad P(B \mid A) = \frac{41}{79}$$

$$P(A \cap B) = P(A)P(B \mid A) = \frac{21}{40} \cdot \frac{41}{79} \approx 0.272$$

	Grade		
	A or B	C or below	Totals
Completed	34	8	42
Did Not Complete	12	26	38
Totals	46	34	80

Example

- A batch of 5 computers has 2 faulty computers. If the computers are chosen at random (without replacement), what is the probability that the first two inspected are both faulty?

Use $P(A \cap B) = P(A)P(B|A)$

P(first computer faulty AND second computer faulty)

$$= \underbrace{\text{P(first computer faulty)}}_{\frac{2}{5}} \times \underbrace{\text{P(second computer faulty | first computer faulty)}}_{\frac{1}{4}}$$

$$= \frac{2}{5} \times \frac{1}{4} = \frac{2}{20} = \frac{1}{10}$$

Card example

- Drawing two random cards from a pack without replacement, what is the probability of getting two hearts? [13 of the 52 cards in a pack are hearts]

Answer:

$$\frac{13 \cdot 12}{52 \cdot 51} \approx 0.059$$

Example – 1

TV screens produced by a manufacturer have defects 10% of the time. An automated mid-production test is found to be 80% reliable at detecting faults (if the TV has a fault, the test indicates this 80% of the time, if the TV is fault-free there is a false positive only 20% of the time). If a TV fails the test, what is the probability that it has a defect?

1. What is the probability that a random TV fails the test?
2. Given that a random TV has failed the test, what is the probability it is because it has a defect?

Example – 2

- Let D="TV has a defect"
- Let F="TV fails test"
- The question tells us: $P(D) = 0.1$, $P(F|D) = 0.8$
- Two independent ways to fail the test:

  TV has a defect and test shows this, -OR- TV is OK but get a false positive

$$P(F|D^c) = 0.2$$

$$P(F) = P(F \cap D) + P(F \cap D^c)$$

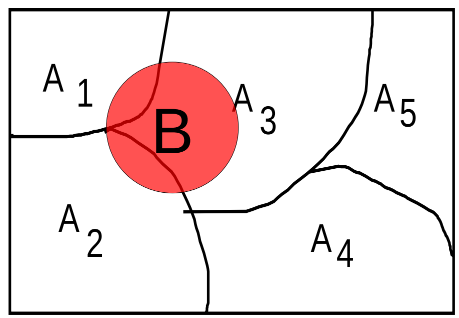
$$= P(F|D)P(D) + P(F|D^c)P(D^c)$$

$$= 0.8 \times 0.1 + 0.2 \times (1 - 0.1) = 0.26$$

Total Probability Rule

$$P(F) = P(F \cap D) + P(F \cap D^c) = P(F|D)P(D) + P(F|D^c)P(D^c)$$

- If $A_1, A_2 \ldots, A_k$ form a partition (a mutually exclusive list of all possible outcomes) and B is any event then
- $P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_k)P(A_k) = \sum_k P(B|A_k)P(A_k)$



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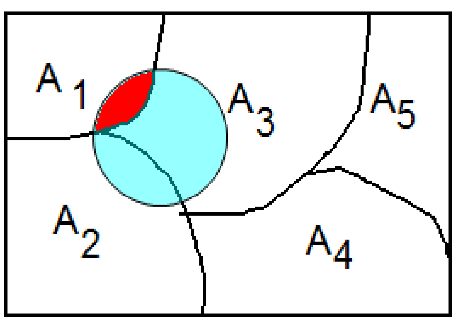

$P(A_1 \cap B) = P(B|A_1)P(A_1)$

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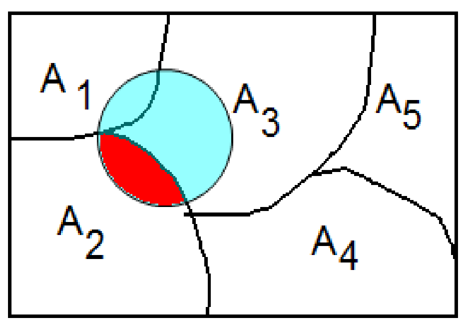

$P(A_2 \cap B) = P(B|A_2)P(A_2)$

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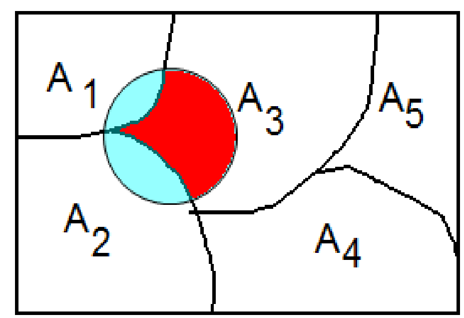
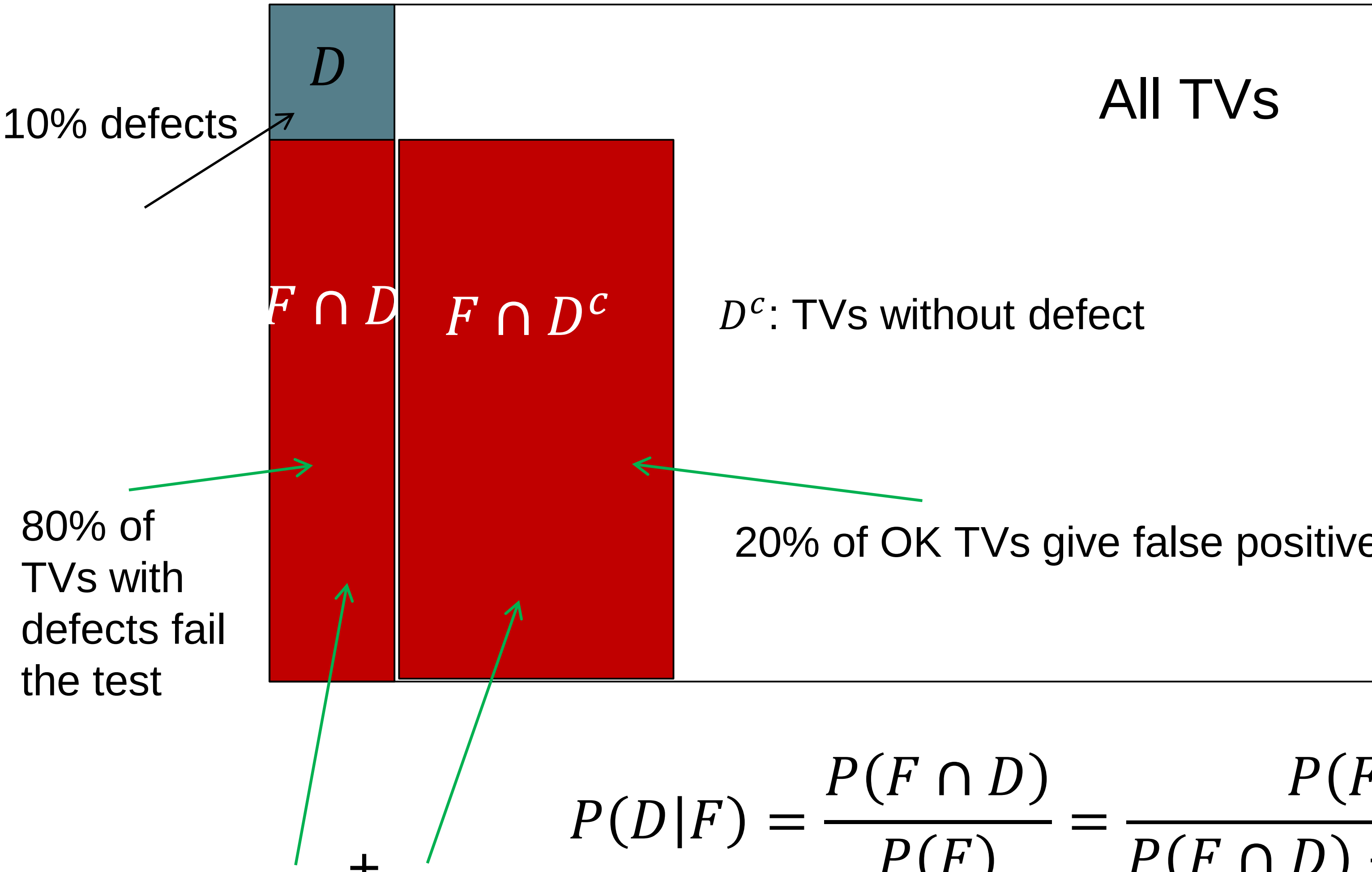


$P(A_3 \cap B) = P(B|A_3)P(A_3)$

TV defects

- Let D="TV has a defect"
- Let F="TV fails test"
- We previously showed using the total probability rule that
- $P(F) = P(F|D)P(D) + P(F|D^c)P(D^c) = 0.8 \times 0.1 + 0.2 \times (1 - 0.1) = 0.26$
- *When we get a test fail, what fraction of the time is it because the TV has a defect?*

Solution



F: TVs that fail the test

$$P(D|F) = \frac{P(F \cap D)}{P(F)} = \frac{P(F \cap D)}{P(F \cap D) + P(F \cap D^c)}$$

Answer

- Let D="TV has a defect"
- Let F="TV fails test"

We previously showed using the total probability rule that

$$P(F) = P(F|D)P(D) + P(F|D^c)P(D^c)$$
$$= 0.8 \times 0.1 + 0.2 \times (1 - 0.1) = 0.26$$

- *When we get a test fail, what fraction of the time is it because the TV has a defect?*

$$P(D|F) = \frac{P(D \cap F)}{P(F)}$$

- Know $P(F|D) = 0.8, P(D) = 0.1$:

$$P(D|F) = \frac{P(F|D)P(D)}{P(F)} = \frac{0.8 \times 0.1}{0.26} \approx 0.3077$$

Conditional Probability Example

- A study was carried out to investigate the link between people's lifestyles and cancer. One of the areas looked at was the link between lung cancer and smoking. 10,000 people over the age of 55 were studied over a 10 year period. In that time 277 developed lung cancer.

	Cancer	No Cancer	Total
Smoker	241	3,325	3,566
Non-Smoker	36	6,398	6,434
Total	277	9,723	10,000

What is the likelihood of somebody developing lung cancer given that they smoke?

Conditional Probability Example

Event A: A person develops lung cancer

Event B: A person is a smoker

P(A) = 277/10,000 = 0.027 P(B) = 3,566/10,000 = 0.356

$$P(A \cap B) = 241/10{,}000 = 0.0241$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.0241}{0.3566} = 0.068$$

	Cancer	No Cancer	Total
Smoker	241	3,325	3,566
Non-Smoker	36	6,398	6,434
Total	277	9,723	10,000

Bayes' Theorem

Definition. Let Ω be the sample space of a random experiment and let A_1 and A_2 be two events such that

$$A_1 \cup A_2 = \Omega \quad \text{and} \quad A_1 \cap A_2 = \varnothing$$

The collection of sets $\{A_1, A_2\}$ is called a *partition* of Ω.

Theorem. If A_1 and A_2 form a partition of Ω, and $B \subset \Omega$ is any event, then for $i = 1, 2$

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{\underbrace{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)}_{P(B)}}$$

Example

- Medical tests for the presence of drugs are not perfect. They often give *false positives* where the test indicates the presence of the drug in a person who has not used the drug, and *false negatives* where the test does not indicate the presence of the drug in a person who has actually used the drug.
- Suppose a certain marijuana drug test gives 13.5% false positives and 2.5% false negatives and that in the general population 0.5% of people actually use marijuana. If a randomly selected person from the population tests positive, find the conditional probability that the person actually used marijuana.

Example – 2

Define the events

$$U = \text{used marijuana}, \ \ NU = \text{not used marijuana},$$
$$T^{+} = \text{tested positive}, \ \ \text{and } T^{-} = \text{tested negative}$$

Note that {*U*, *NU*} forms a partition.

$$P\left(T^{+} \mid NU\right) = 0.135, \quad P\left(T^{-} \mid NU\right) = 0.865,$$
$$P\left(T^{-} \mid U\right) = 0.025, \quad P\left(T^{+} \mid U\right) = 0.975$$

Example – 3

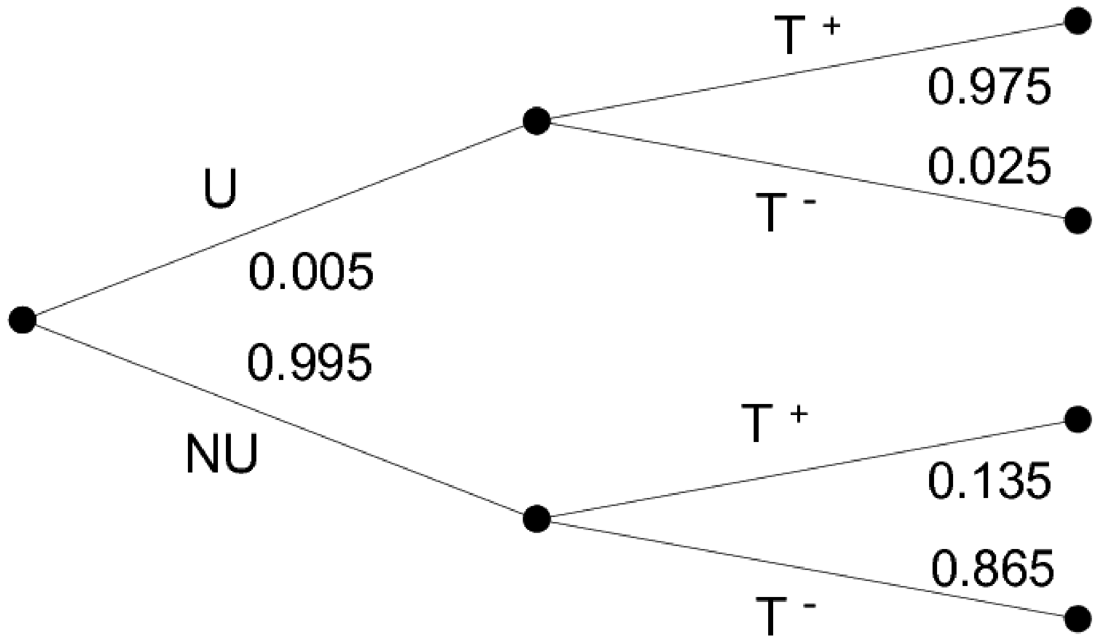
$$P\left(U \mid T^{+}\right)=\frac{P(U)\,P\left(T^{+} \mid U\right)}{P(U)\,P\left(T^{+} \mid U\right)+P(NU)\,P\left(T^{+} \mid NU\right)}$$

$$=\frac{(0.005)(0.975)}{(0.005)(0.975)+(0.995)(0.135)}$$

$$\approx 0.035$$

Example – 4

- “Tree diagram”



$$P\left(U \mid T^{+}\right)=\frac{(0.005)(0.975)}{(0.005)(0.975)+(0.995)(0.135)}$$

Example: Evidence in court

- The cars in a city are 90% black and 10% grey. A witness to a bank robbery briefly sees the escape car, and says it is grey. Testing the witness under similar conditions shows the witness correctly identifies the colour 80% of the time (in either direction). What is the probability that the escape car was actually grey?

Answer: Let G = car is grey, B=car is black, W = Witness says car is grey.

Bayes' Theorem

$$P(G|W) = \frac{P(W \cap G)}{P(W)} = \frac{P(W|G)P(G)}{P(W)}.$$

Use total probability rule to write

$$P(W) = P(W|G)P(G) + P(W|B)P(B) = 0.8 \times 0.1 + 0.2 \times 0.9 = 0.26$$

Hence:

$$P(G|W) = \frac{P(W|G)P(G)}{P(W)} = \frac{0.8 \times 0.1}{0.26} \approx 0.31$$

Failing a drugs test

A drugs test for athletes is 99% reliable: applied to a drug taker it gives a positive result 99% of the time, given to a non-taker it gives a negative result 99% of the time. It is estimated that 1% of athletes take drugs.

1. What fraction of randomly tested athletes fail the test?

- Let F="fails test"
- Let D="takes drugs"

Question tells us

$$P(D) = 0.01, P(F|D) = 0.99,\ P(F|D^c) = 0.01$$

From total probability rule:

$$P(F) = P(F|D)P(D) + P(F|D^c)P(D^c) = 0.99 \times 0.01 + 0.01 \times 0.99 = 0.0198$$

i.e. 1.98% of randomly tested athletes fail

Failing a drugs test

A drugs test for athletes is 99% reliable: applied to a drug taker it gives a positive result 99% of the time, given to a non-taker it gives a negative result 99% of the time. It is estimated that 1% of athletes take drugs.
2. A random athlete has failed the test. What is the probability the athlete takes drugs?

Let F="fails test"
Let D="takes drugs"
Question tells us

$$P(D) = 0.01, P(F|D) = 0.99,\ P(F|D^c) = 0.01$$

Bayes' Theorem gives

$$P(D|F) = \frac{P(F|D)P(D)}{P(F)}$$

We need $P(F) = P(F|D)P(D) + P(F|D^c)P(D^c) = 0.99 \times 0.01 + 0.01 \times 0.99 =$ 0.0198

Hence: $P(D|F) = \frac{P(F|D)\,P(D)}{P(F)} = \frac{0.99\times 0.01}{0.0198} = \frac{0.0099}{0.0198} = \frac{1}{2}$

Review

Conditional probability

General definition: under assumption that P(B) > 0 (event B has to be possible)

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Total Probability Rule

$$P(F) = P(F \cap D) + P(F \cap D^c) = P(F|D)P(D) + P(F|D^c)P(D^c)$$

- If $A_1, A_2 \dots, A_k$ form a partition (a mutually exclusive list of all possible outcomes) and B is any event then
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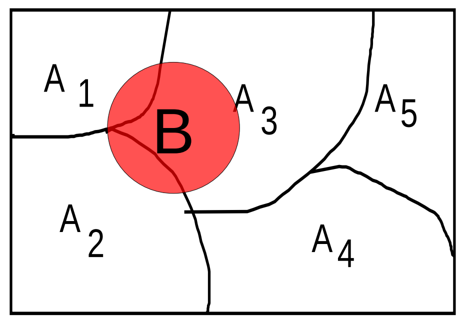

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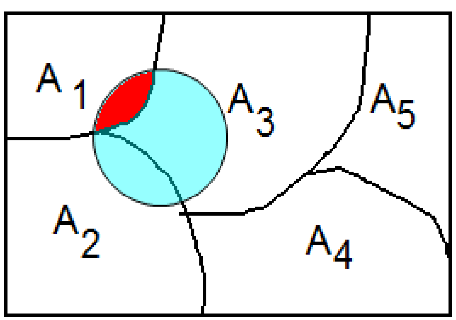

$P(A_1 \cap B) = P(B|A_1)P(A_1)$

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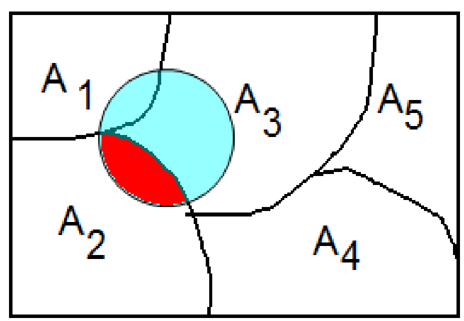

$P(A_2 \cap B) = P(B|A_2)P(A_2)$

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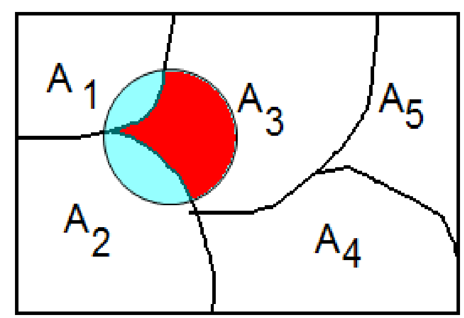

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$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{\underbrace{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)}_{P(B)}}$$

Thank you for attention!