Kyiv
School of
Economics

Theory of Probabilies

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Outline

1. Definitions of probability
2. Random and elementary events; sample space
3. Relation of events

Definitions of probability

- Classical
- Geometric
- Frequency (von Mises)
- Axiomatic (Kolmogorov)

Classical definition of probability

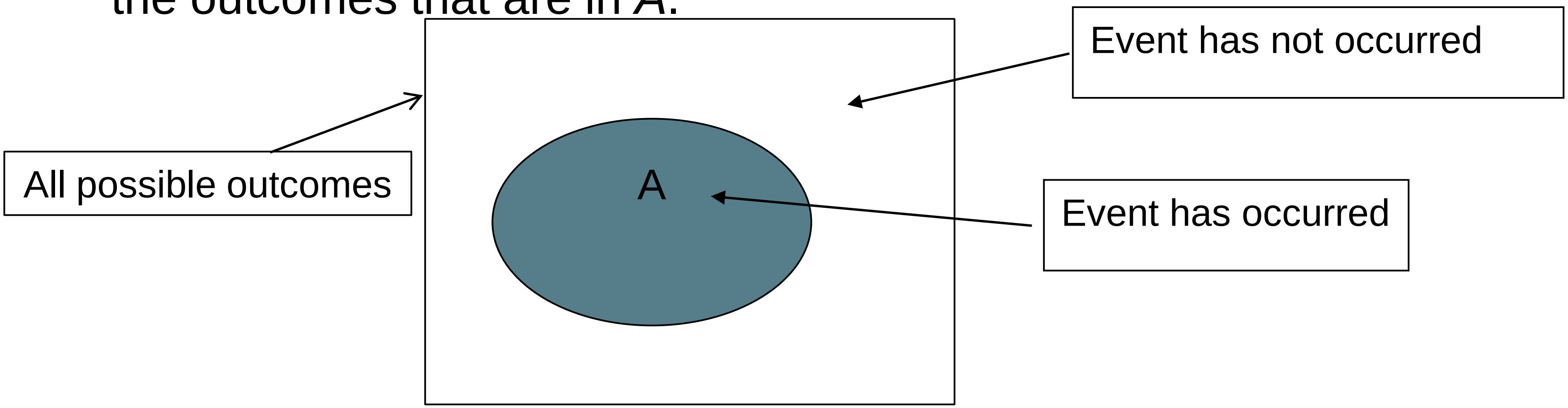
- First (classical) definition of probability was formulated by P.S. Laplace in 1812.
- Consider random experiment that results always in exactly one of N equally possible results.
- Probability of event A is given as a ratio of number n_a of outcomes favorable to A to the number of all possible outcomes N

$$P(A) = \frac{n_a}{N}$$

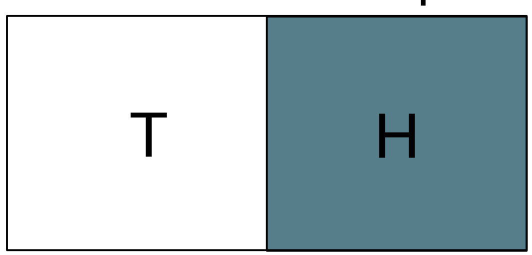
A is a subset of a sure event Ω: $A \subset \Omega$

Example

- If there a fixed number of equally likely outcomes $P(A)$ is the fraction of the outcomes that are in *A*.



E.g. for a coin toss there are two possible outcomes, Heads or Tails

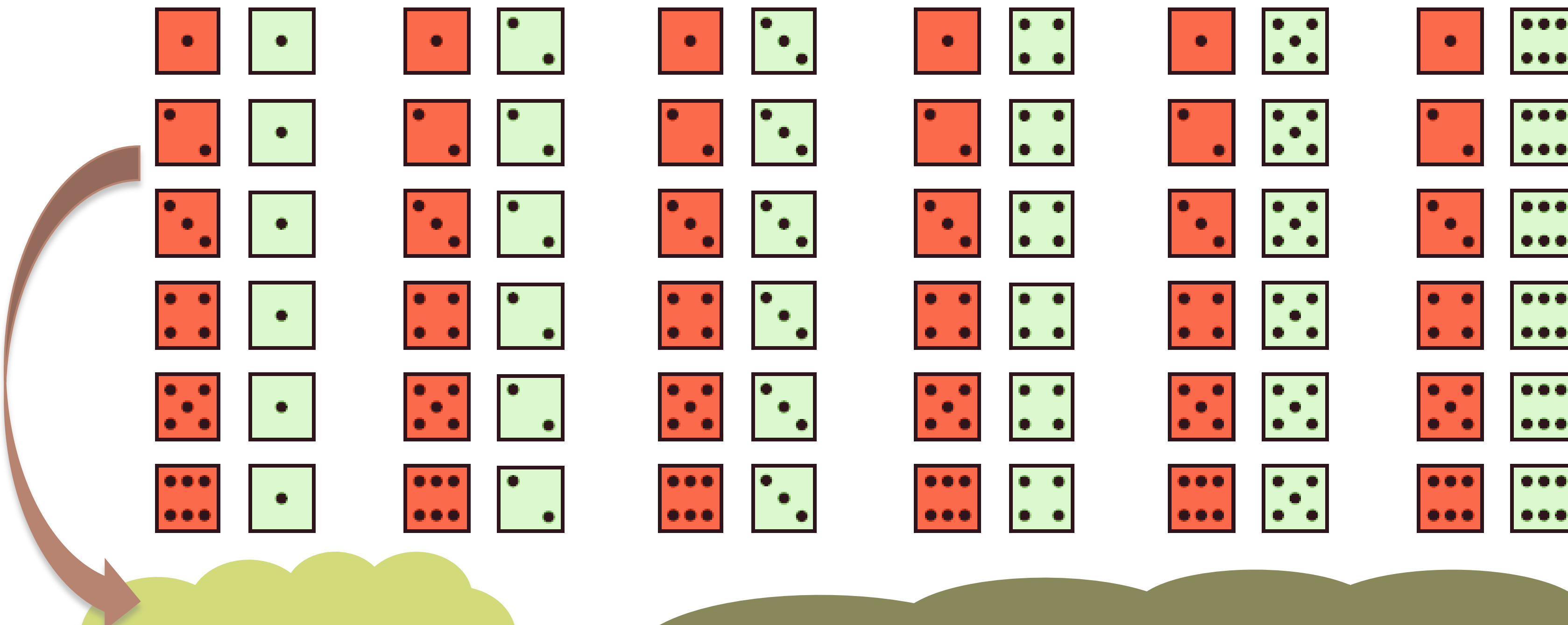


P(*result of a coin toss is heads*) = 1/2.

Intuitive idea: P(*A*) is the typical fraction of times *A* would occur if an experiment were repeated very many times.

Example

- Give a probability model for the chance process of rolling two fair, six-sided dice—one that's red and one that's green.



Sample Space
36 Outcomes

Since the dice are fair, each outcome is equally likely.
Each outcome has probability 1/36.

Geometric definition of probability

- Introduced in order to treat the cases of infinite number of outcomes.
- Consider that in r-dimensional space where there exists a region G that contains a smaller region g. A random experiment consists in a random choice of a point in G assuming that all points are equally probable.
- Probability of event A that randomly chosen point will be found in a region g is given as

$$P(A) = \frac{measure\,(g)}{measure\,(G)}$$

Frequency definition of probability

- Proposed by R. von Mises in 1931. Has no drawbacks of classical nor geometric definition. Is intuitive and agrees with the observed laws concerning frequency. However, it is unacceptable as a definition of mathematical quantity (a posteriori).
- Probability of event A is a limit of frequency of this event when the number of experiments n tends to infinity

$$P(A) = \lim_{n \to \infty} \frac{n(A)}{n}$$

Axiomatic definition of probability

To each random event A we ascribe a number P(A), named a probability of this event that satisfies the following axioms:

1. $0 \leq P(A) \leq 1$.

2. Probability of a sure event equals to 1

$$P(\Omega) = 1$$

3. (Countable additivity of probability) Probability of an alternative of countable disjoint (mutually exclusive) events is equal to the sum of probabilities of these events: if $A_1, A_2, \ldots \subset \Omega$, while for each pair of i, j ($i \neq j$) the following condition is fulfilled $A_i \cap A_j = \emptyset$, then

$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$$

Example

- **Experiment**: Toss two coins and observe the up face on each
- **Probability of each event:**

1. E = *HH* => P(HH) = 1/4
2. E = *HT* => P(HT) = 1/4
3. E = *TH* => P(TH) = 1/4
4. E = *TT* => P(TT) = 1/4

$$\sum_{i=1}^{k} P(E_i) = 1$$

Example

- Bag with 4 blue, 3 red, 2 green, and 1 yellow cube

- **Question:** “How likely is it that we get a green cube?”
  - Let G = event we get a green cube
  - We want to know $P(G)$

Theoretical Approach

Let

$$\Omega = \{B_1, B_2, B_3, B_4, R_1, R_2, R_3, G_1, G_2, Y_1\}, \text{ and}$$

then $G = \{G_1, G_2\},$

$$n(\Omega) = 10, \quad n(G) = 2$$

$$\Rightarrow P(G) = \frac{2}{10} = 0.2$$

Relative Frequency

- Choose a cube, record its color, replace, and repeat 50 times

Student 1

Color	Freq	Rel Freq
Blue	19	0.38
Red	15	0.30
Green	12	0.24
Yellow	4	0.08

Student 2

Color	Freq	Rel Freq
Blue	21	0.42
Red	16	0.32
Green	7	0.14
Yellow	6	0.12

Law of Large Numbers

- Combine students' results

  50 + 50 = 100 trials

  12 + 7 = 19 green cubes

$$\text{Rel Freq} = \frac{19}{100} = 0.19 \approx 0.2$$

Law of large numbers!

Random or elementary events

- For each random experiment we consider a set of its all possible outcomes, i.e., sample space Ω. These outcomes are called random events.
- Among all random events we can distinguish some simple, irreducible ones that are characterized by a single outcome. These are elementary events.

Example:

- All sets {k}, where k ∈ N if objects are being counted and the sample space is Ω = {0, 1, 2, 3, ...} (the natural numbers).

Example of a random events

A coin is tossed twice. Possible outcomes are as follows:

- (T, T) – both tails
- (H, T) – head first, tail next
- (T, H) – tail first, head next
- (H, H) – both heads

- Ω={(T, T); (H, T) ; (T, H); (H, H)} is a set of elementary events, i.e., the sample space

- If the set of elementary events contains n-elements then the number of all random events is 2n

Example of a random event

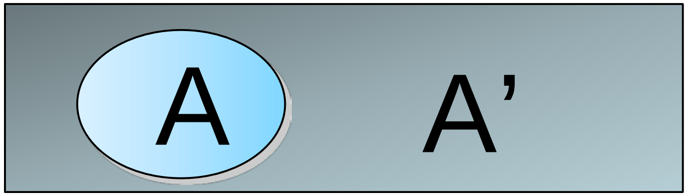
- A = {(T,T); (T,H); (H,T)} – at least one tail T
- B = {(T,H); (T,T)} – tail in the first two essays
- G = {(T,T)} – both tails
- H = {(T,H); (H,T)} – exactly one tail
- …

Example for individual study

- Count all random events (including sure and impossible ones) in the experiment that consists in throwing a dice. Determine the space of events.

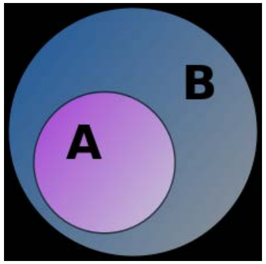
Relations of events – 1

- Complementary event – event A does not take place A'



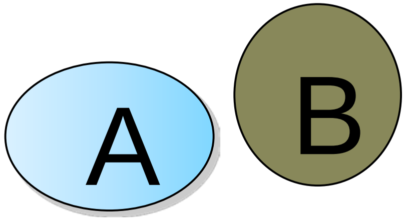
- Event A incites B (subset A is totally included in B)

$$A \subset B$$



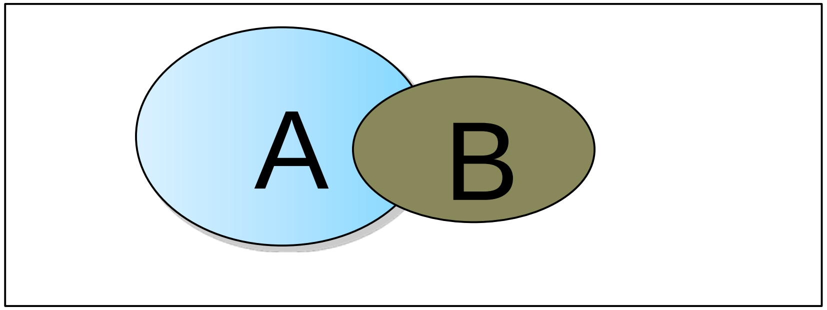
- Events A and B are mutually exclusive

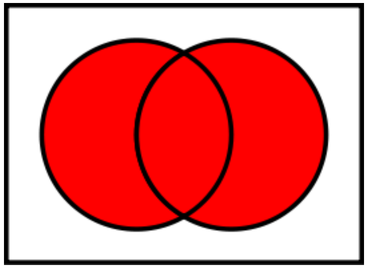
$$A \cap B = \varnothing$$



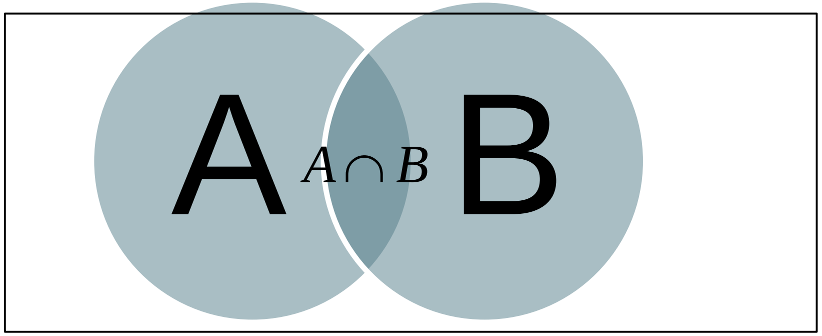
Relations of events – 2

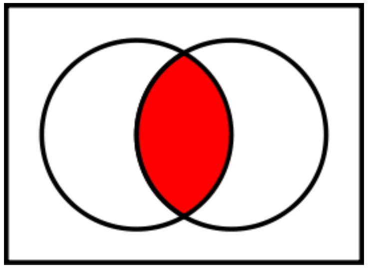
- Sum of events – when at least one of events A or B takes place (union of sets)





- Product of events – both A and B happen (intersection of sets)



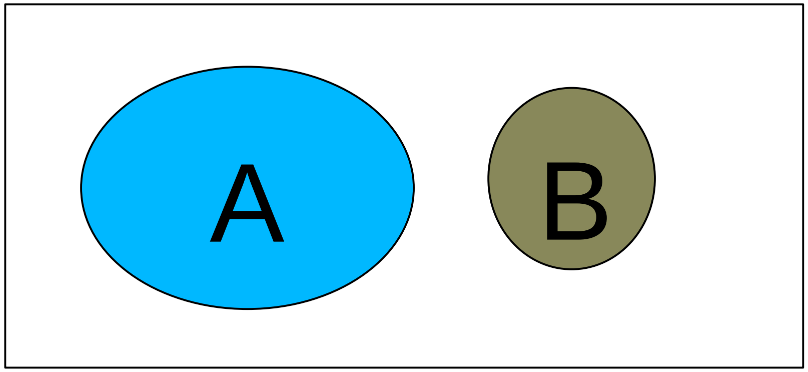


Consequences of axioms

Probability of sum of the mutually exclusive random events A i B equals to the sum of probabilities of these events

(Kolmogorov, 1933)

$$P(A \cup B) = P(A) + P(B), where \quad A \cap B = \varnothing$$



Probability

- The probability of an event A is equal to the sum of all the probabilities in event A.

Experiment: Toss two coins and observe the up face on each

- Event A: {Observe exactly one head}

  $P(A) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

- Event B : {Observe at least one head}

  $P(B) = P(HH) + P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$

Self-task

- Calculate probability *P(A)* that sum of throwing 2 fare dice is less then 7.

Sample Space

	Second Die					
First Die	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$$P(A) = \frac{15}{36} = \frac{5}{12} \approx 0.417$$

Random Variable

- Let *X* = the sum of the dice
  - Called a *random variable*

x	2	3	4	5	6	7	8	9	10	11	12
Num Outcomes	1	2	3	4	5	6	5	4	3	2	1
$P(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

  - Table is called the *distribution of X*

Example

- Suppose we randomly choose *any* number between 1 and 100
- Let

$$\Omega = [1, 100] \quad \text{and} \quad B = [75, 100]$$

then a reasonable assignment of probability is

$$P(B) = \frac{\text{length of interval } B}{\text{length of interval } \Omega} = \frac{26}{100} = 0.26$$

Questions

1. What is the sample space when a coin is tossed 3 times?
2. What is the sample space for the number of aces in a hand of 13 playing cards?
3. If a fair die is thrown what is the probability of throwing a prime number?
4. If two fair dice are thrown what is the probability that at least one score is a prime number?
    1. What is the compliment of the event.
    2. What is its probability?

Questions

A factory has two assembly lines, each of which is shut down (S), at partial capacity (P), or at full capacity (F). The following table gives the sample space

Event A	P(A)	Event A	P(A)	Event A	P(A)
(S,S)	0.02	(S,P)	0.06	(S,F)	0.05
(P,S)	0.07	(P,P)	0.14	(P,F)	0.20
(F,S)	0.06	(F,P)	0.21	(F,F)	0.19

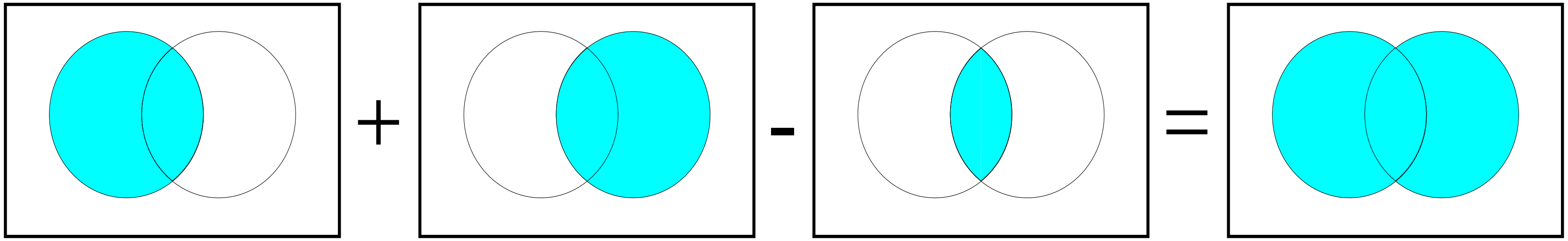
For where (S,P) denotes that the first assembly line is shut down and the second one is operating at partial capacity. What is the probability that:

a) Both assembly lines are shut down?
b) Neither assembly lines are shut down?
c) At least one assembly line is on full capacity?
d) Exactly one assembly line is at full capacity?

Addition Rule

- For any two events A and B,

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



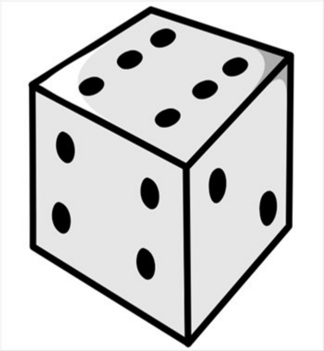
- Note: "*A* or *B*" = $A \cup B$ includes the possibility that both *A* and *B* occur.

Example

- Throwing a fair dice, let events be

  A = get an odd number
  B = get a 5 or 6

  What is P(A or B)?



$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= P(\text{odd}) + P(5 \text{ or } 6) - P(5)$$
$$= \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

This is consistent since $P(A \cup B) = P(\{1,3,5,6\}) = \frac{4}{6} = \frac{2}{3}$

Complementary Event

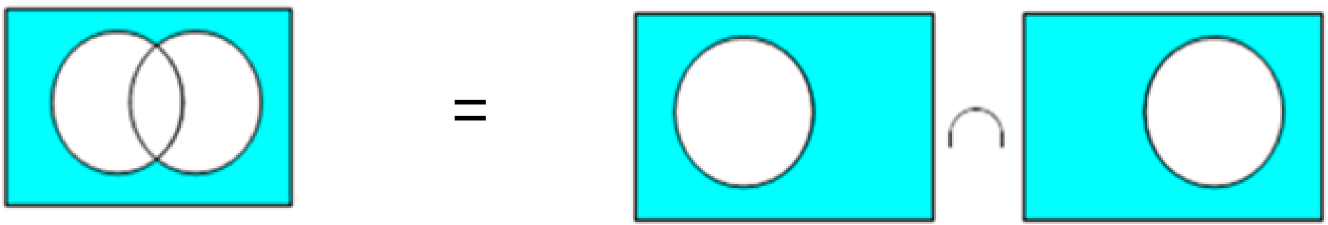
- The complementary of an event A is the event that A does not occur denoted by A´ (A^c)

- Note that A U A` = Ω, the sample space

- P(A) + P(A`) =1 => P(A) = 1 – P(A`)

Sum of probabilities

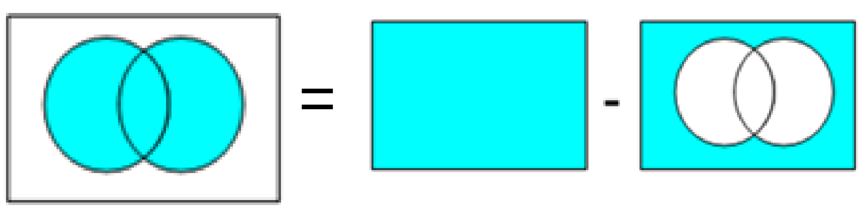
- "Probability of not getting either A or B = probability of not getting A and not getting B"



$(A \cup B)^c$ A^c B^c

$$(A \cup B)^c = \mathrm{A^c} \cap B^c$$

Complements Rule $\Rightarrow P(A \cup B) = 1 - P(A^c \cap B^c)$



i.e. P(A or B) = 1 – P("not A" and "not B")

Example

- Throwing a fair dice, let events be

  A = get an odd number

  B = get a 5 or 6

  What is P(A or B)?

- Alternative answer

  A^c={2,4,6}, B^c = {1,2,3,4} so $A^c \cap B^c$ ={2,4}.

- Hence

$$P(A \text{ or } B) = 1 - P(A^c \cap B^c) = 1 - P(\{2,4\}) = 1 - \frac{1}{3} = \frac{2}{3}$$

Rules of probability recap

- **Complements Rule:** $\mathbf{P(A^c) = 1 - P(A)}$
  Q. What is the probability that a random card is not the ace of spades?
  A. 1-P(ace of spades) = 1-1/52 = 51/52

- **Multiplication Rule:** $\mathbf{P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)}$
  Q What is the probability that two cards taken (without replacement) are both Aces?
  A $P(\text{first ace})P(\text{second ace}|\text{first ace}) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$

- **Addition Rule:** $\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$
  Q What is the probability of a random card being a diamond or an ace?
  A $P(\text{diamond}) + P(\text{ace}) - P(\text{diamond and ace}) = \frac{1}{4} + \frac{1}{13} - \frac{1}{52} = \frac{4}{13}$

Independent Events

Definition. Two events *A* and *B* are said to be *independent* if

$$P\left(A \cap B\right) = P(A)P(B)$$

If they are not independent, they are said to be *dependent*.

Informally: Independence means the occurrence of one event does not affect the probability of the other event occurring

Self-task

There are three common ways for a system to experience problems, with independent probabilities over a year

A = *overheats*, $P(A)=1/3$

B = *subcomponent malfunctions*, $P(B) = 1/3$

C = *damaged by operator*, $P(C) = 1/10$

What is the probability that the system has one or more of these problems during the year?

Answer: 3/5

Example – 1

Suppose a student has an 8:30 AM statistics test and is worried that her alarm clock will fail and not ring, so she decides to set two different battery-powered alarm clocks. If the probability that each clock will fail is 0.005. Find the probability that at least one clock will ring.

- Let F_1 and F_2 be the events that the first and second clocks fail
- Assume independence

Example – 2

$$
\begin{aligned}
P(\text{at least one rings}) &= 1 - P(\text{both fail}) \\
&= 1 - P\left(F_1 \cap F_2\right) \\
&= 1 - P\left(F_1\right)P\left(F_2\right) \\
&= 1 - \left(0.005\right)\left(0.005\right) \\
&= 0.999975
\end{aligned}
$$

Mutual Independence

Definition. Three events *A*, *B*, and *C* are said to be *pairwise independent* if

$$P(A \cap B) = P(A)P(B), \;\; P(A \cap C) = P(A)P(C),$$
$$\text{and} \;\; P(B \cap C) = P(B)P(C)$$

They are said to be *mutually independent* (or simply *independent*) if they are pairwise independent and

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

Example – 1

Suppose that on a college campus of 1,000 students (referred to as the *population*), 600 support the idea of building a new gym and 400 are opposed. The president of the college randomly selects five different students and talks with each about their opinion. Find the probability that they all oppose the idea.

Example – 2

Let

- A_1 = event that the first student opposes the idea,
- A_2 = event that the second student opposes it, etc.

These events are dependent since the selections are made *without replacement*

$$P\left(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5\right) = \frac{400}{1000} \cdot \frac{399}{999} \cdot \frac{398}{998} \cdot \frac{397}{997} \cdot \frac{396}{996}$$

$$\approx 0.010087$$

Example – 3

Now suppose the selections are made *with replacement*

- The events are independent

$$P\left(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5\right) = \frac{400}{1000} \cdot \frac{400}{1000} \cdot \frac{400}{1000} \cdot \frac{400}{1000} \cdot \frac{400}{1000}$$

$$= (0.4)^5 = 0.01024.$$

Example – 4

- Without replacement: Prob ≈ 0.010087
- With replacement: Prob = 0.01024

- Practically the same!

- **5% Guideline**: If no more than 5% of the population is being selected, then the selections may be treated as independent, **even though they are technically dependent**

REVIEW

Definition of probability

- Classical definition of probability $P(A)=\frac{n_a}{N}$
- Geometric definition of probability $P(A)=\frac{measure(g)}{measure(G)}$
- Frequency definition of probability $P(A)=\lim_{n\to\infty}\frac{n(A)}{n}$

Axiomatic definition of probability

To each random event A we ascribe a number P(A), named a probability of this event that satisfies the following axioms:

1. $0 \leq P(A) \leq 1$.

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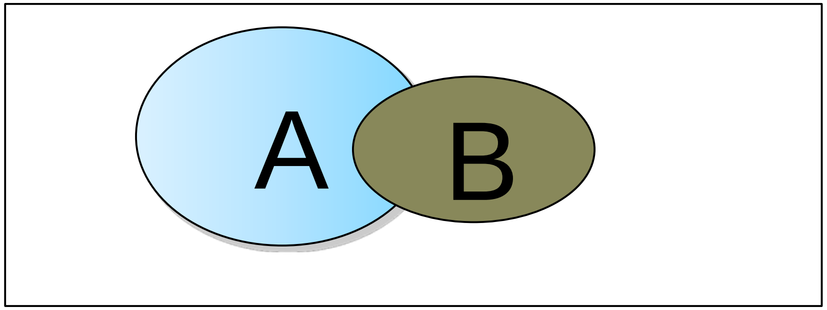
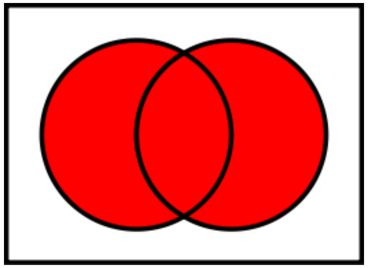
$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$$

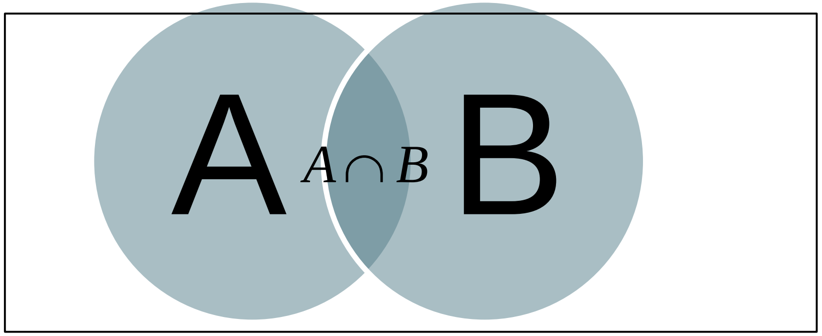
Relations of events – 1

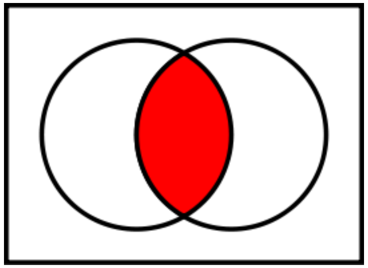
- Sum of events – when at least one of events A or B takes place (union of sets)





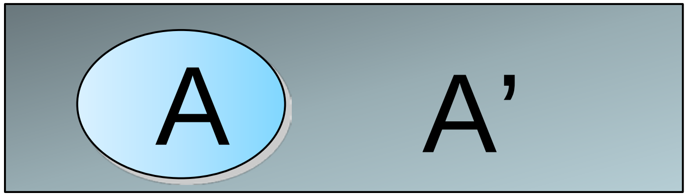
- Product of events – both A and B happen (intersection of sets)





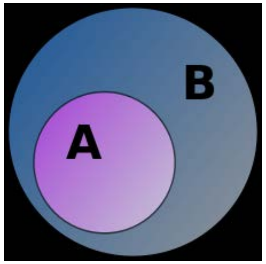
Relations of events – 2

- Complementary event– event A does not take place



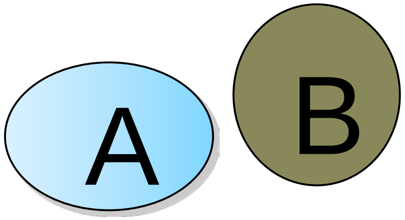
- Event A incites B (subset A is totally included in B)

$$A \subset B$$



- Events A and B are mutually exclusive

$$A \cap B = \varnothing$$



Thank you for attention!

KSE