Kyiv School of Economics

Elements of combinatorics

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Outline

1. Combinatorics and counting problems: sum rule and product rule
2. Factorial
3. Permutation, combination, arrangements

Introduction to combinatorics and counting problems

- Combinatorics concerns itself with finite collections of discrete objects. With the growth of digital devices, especially digital computers, discrete mathematics has become more and more important.

- Counting problems arise when the combinatorial problem is to count the number of different arrangements of collections of objects of a particular kind. Such counting problems arise frequently when we want to calculate probabilities and so they are of wider application than might appear at first sight. Some counting problems are very easy, others are extremely difficult.

Problem I: A café menu

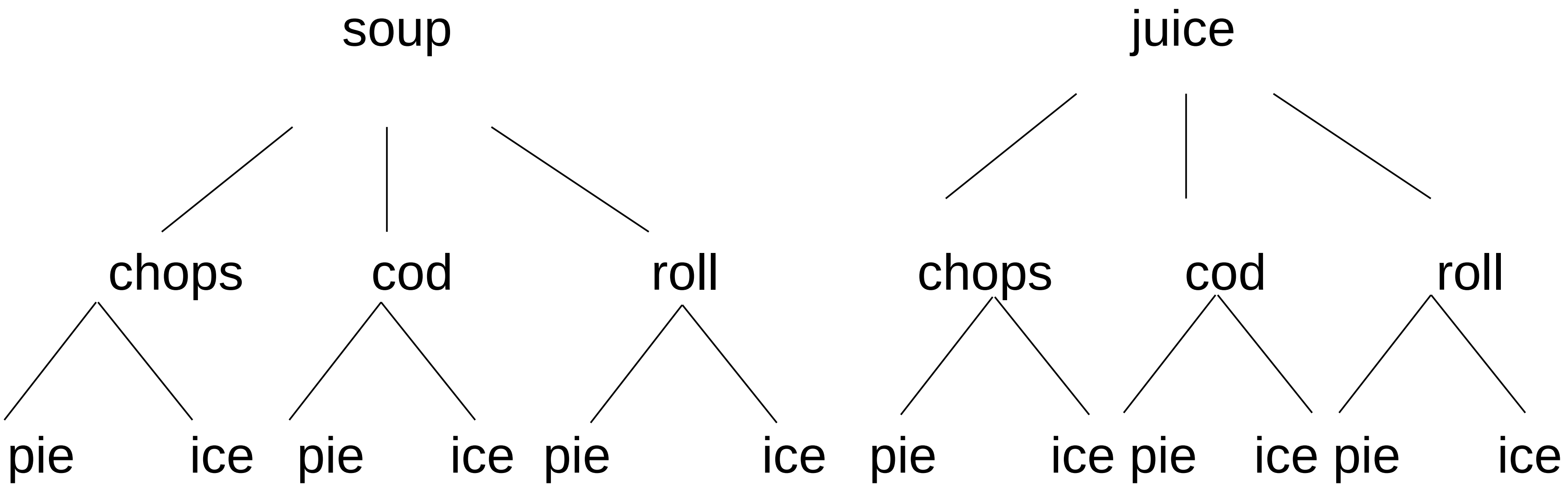
- Tomato soup
- Fruit juice

- Lamb chops
- Baked cod
- Nut roll

- Apple pie
- Strawberry ice

How many different three course meals could you order?

Solution to problem I



We would obtain 2x3x2=12 as the total of possible meals.

Problem II: horse race

In a race with 20 horses, in how many ways the first three places can be filled?

Solution

There are 20 horses that can come first. Whichever horse comes first, there are 19 horses left that can come second. So there are 20x19=380 ways in which the first two places can be filled. In each of these 380 cases there are 18 horses which can come third. So there are:

20x19x18=380x18=6840 ways in which the first three positions can be filled.

What is a difference between these two problems?

Counting problems

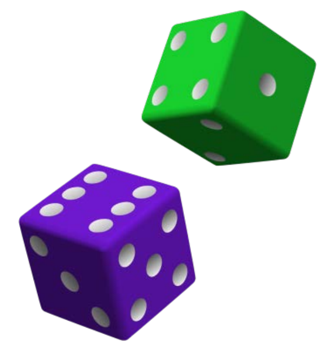
In many situations it is necessary to determine the number of elements of the set under considerations.

We use simple arithmetic methods:

- sum rule
- product rule

coin toss

drawing cards from a deck



dice throw

Sum Rule

If two events are mutually exclusive, that is, they cannot occur at the same time, then we must apply the sum rule.

Theorem: If an event e_1 can be realized in n_1 ways, an event e_2 in n_2 ways, and e_1 and e_2 are **mutually exclusive** then the number of ways of both events occurring is $n_1 + n_2$

There is a natural generalization to any sequence of m tasks; namely the number of ways m mutually exclusive events can occur

$$n_1 + n_2 + \dots + n_{m-1} + n_m$$

Principle of Inclusion-Exclusion (PIE)

- Say there are two events, e_1 and e_2, for which there are n_1 and n_2 possible outcomes respectively.
- Now, say that only one event can occur, not both.
- In this situation, we cannot apply the sum rule. Why?

- … because we would be overcounting the number of possible outcomes.
- Instead we have to count the number of possible outcomes of e_1 and e_2 minus the number of possible outcomes in common to both; i.e., the number of ways to do both tasks.

Product Rule

If two events are not mutually exclusive (that is we do them separately), then we apply the product rule

Theorem: Suppose a procedure can be accomplished with two disjoint subtasks. If there are n_1 ways of doing the first task and n_2 ways of doing the second task, then there are

$n_1 \times n_2$ ways of doing the overall procedure

Application of sum and product rules

- There are two towers at the entrance to the castle. The first is protected by a two-digits „even” code while the second by a two-digits „odd” code. It is sufficient to break one code in order to enter. How many ways there are to the castle?

- Even code. Possible tens: 2,4,6,8
  Possible units: 0,2,4,6,8
  Product rule: $5 \times 4 = 20$

- Odd code. Possible tens: 1,3,5,7,9
  Possible units: 1,3,5,7,9
  Product rule $5 \times 5 = 25$

- Sum rule: $25+20=45$

Factorial

- **Definition.** Let n > 0 be an integer. The symbol n! (read "n factorial") is

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

For convenience, we define 0! = 1.

Permutation vs combination

- Ordered arrangement (sequence) = permutation

  (1,2,3); (2,1,3); (3,1,2) etc.

- Order is not important (set, subset) = combination

  {1,2,3}

In both cases we have to distinguish: with or without **replacement**

Permutations

Definition. Suppose we are choosing r objects from a set of n objects and these requirements are met:

- *The n objects are all different.*
- *We are choosing the r objects without replacement.*
- *The order in which the choices are made is important.*

Then the number of ways the overall choice can be made is called the number of permutations of n objects chosen r at a time.

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Permutations-without replacement – 1

- → "Trial and error" method:
- Systematically write out all possibilities:

A B C D E

A B C E D

A B D C E

A B D E C

A B E C D

A B E D C

…

Quickly becomes a pain!

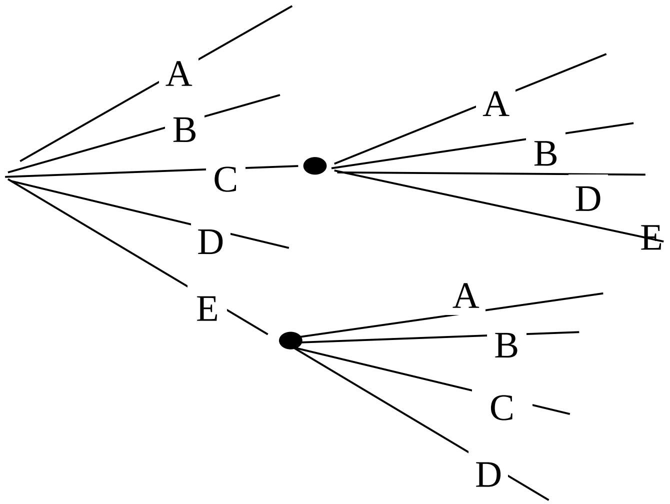
Easier to figure out patterns using a probability tree!

Permutations-without replacement – 2

<u>Seat One:</u>
5 possible

<u>Seat Two:</u>
only 4 possible

Etc….



…….

# of permutations = 5 x 4 x 3 x 2 x 1 = 5!

There are 5! ways to order 5 people in 5 chairs (since a person cannot repeat)

Permutations-without replacement – 3

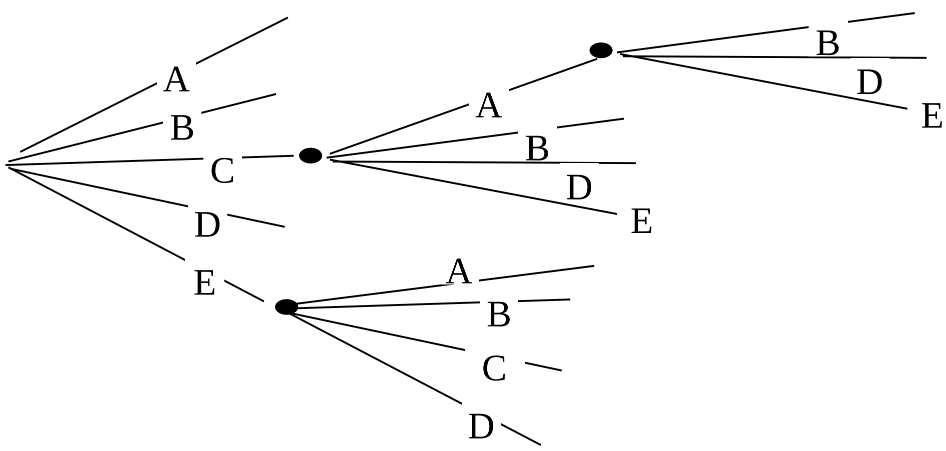
What if you had to arrange 5 people in only 3 chairs (meaning 2 are out)?

$$5 \cdot 4 \cdot 3 =$$

<u>Seat One:</u>
5 possible

<u>Seat Two:</u>
Only 4 possible

<u>Seat Three:</u>
only 3 possible



$$\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{5!}{2!} =$$

$$\frac{5!}{(5-3)!}$$

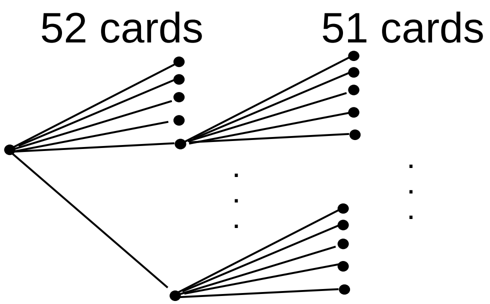
Permutations-without replacement – 4

Note this also works for 5 people and 5 chairs:

$$\boxed{\frac{5!}{(5-5)!} = \frac{5!}{0!} = 5!}$$

Example

- How many two-card hands can I draw from a deck when order matters (e.g., ace of spades followed by ten of clubs is different than ten of clubs followed by ace of spades)



$$\frac{52!}{(52-2)!} = 52 \cdot 51$$

Combinations – 1

Definition. Suppose we are choosing *r* objects from a set of *n* objects and these requirements are met:

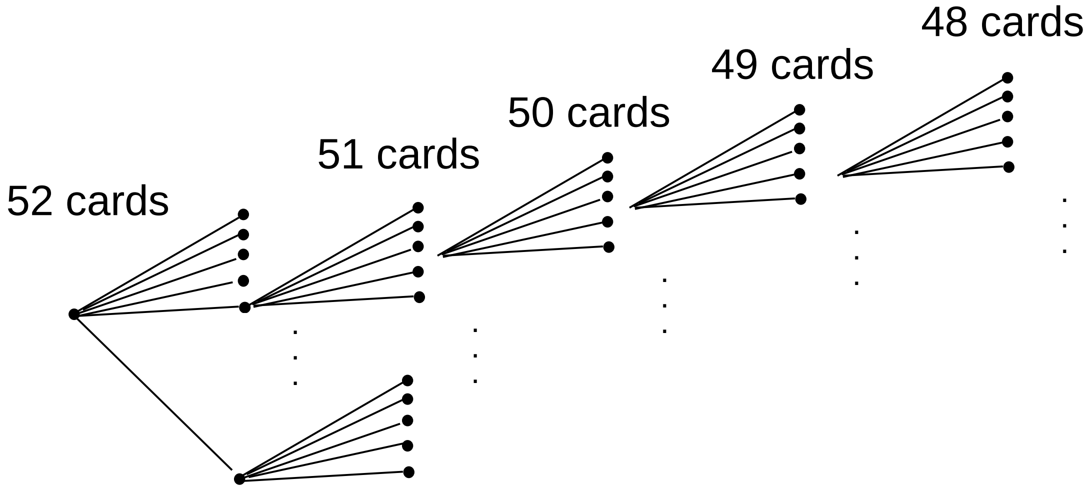
- The *n* objects are all *different*.
- We are choosing the *r* objects *without replacement*.
- The order in which the choices are made is *not* important.

Then the number of ways the overall choice can be made is called the number of combinations of n objects chosen r at a time.

$$_{n}C_{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Combinations – 2

How many five-card hands can I draw from a deck when order does **not** matter?



$$\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{?}$$

Combinations – 3

- How many unique 2-card sets out of 52 cards?

$$\boxed{\frac{52 \cdot 51}{2} = \frac{52!}{(52-2)!2!}}$$

- 5-card sets?

$$\boxed{\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5!} = \frac{52!}{(52-5)!5!}}$$

- r-card sets?

$$\boxed{\frac{52!}{(52-r)!r!}}$$

- r-card sets out of n-cards?

$$\boxed{{}_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!}}$$

Example

A lottery works by picking 6 numbers from 1 to 49. How many combinations of 6 numbers could you choose?

$$ {}_{49}C_6 = \binom{49}{6} = \frac{49!}{43!6!} = 13{,}983{,}816 $$

Which of course means that your probability of winning is 1/13,983,816!

Arrangements

Definition. Suppose we are arranging *n* objects, n_1 are identical, n_2 are identical, … , n_r are identical. Then the *number of unique arrangements of the n objects* is

$$_{n}A_{r}\left(n_1, n_2, \ldots, n_r\right) = \frac{n!}{n_1!\,n_2!\cdots n_r!}$$

Example

15 persons must be divided into 3 groups with 4, 5 and 6 persons respectively.

How many outcomes are possible?

$$ {}_{15}A_3\left(4,5,6\right)=\frac{15!}{4!\,5!\,6!}=630\,630 $$

Compare!

$$ {}_{n}A_2\left(n_1,n-n_1\right)=\frac{n!}{n_1!\left(n-n_1\right)!} \quad \& \quad {}_{n}C_{n_1}=\frac{n!}{n_1!\left(n-n_1\right)!} $$

Example - 1

A local college is investigating ways to improve the scheduling of student activities. A fifteen-person committee consisting of five administrators, five faculty members, and five students is being formed. A five-person subcommittee is to be formed from this larger committee. The chair and co-chair of the subcommittee must be administrators, and the remainder will consist of faculty and students. How many different subcommittees could be formed?

Example – 2

- Two sub-choices:
    1. Choose two administrators.
    2. Choose three faculty and students.

- Number of choices:

$$ {}_5P_2 \times \binom{10}{3} = 20 \times 120 = 2400 $$

Review

Sum and Product Rules

Sum Rule:

If an event e_1 can be realized in n_1 ways, an event e_2 in n_2 ways, and e_1 and e_2 are **mutually exclusive** then the number of ways of both events occurring is $n_1 + n_2$

Product Rule:

Suppose a procedure can be accomplished with two disjoint subtasks. If there are n_1 ways of doing the first task and n_2 ways of doing the second task, then there are $n_1 \times n_2$ ways of doing the overall procedure

Main formulas

- Factorial $$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$
- Permutations $${}_nP_r = \frac{n!}{(n-r)!}$$
- Combinations $${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$
- Arrangements $${}_nA_r(n_1, n_2, \ldots, n_r) = \frac{n!}{n_1! n_2! \cdots n_r!}$$

Thank you for attention!